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THREE-DIMENSIONAL DIGITAL TOPOLOGY: THE GENUS.(U)
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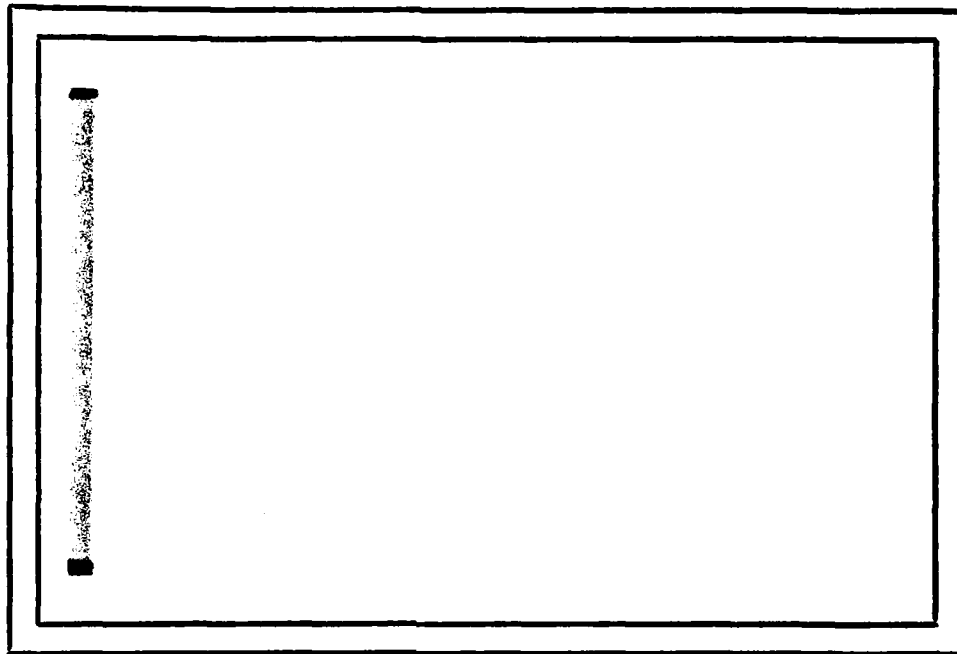
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THREE-DIMENSIONAL
DIGITAL TOPOLOGY: THE GENUS.

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ABSTRACT

This report is one of a series on the geometrical and topological properties of three-dimensional digital images. In this report, we define the genus of a subset of a three-dimensional array and show that the properties of this digital genus are consistent with those of the genus in ordinary (continuous) topology. Since the approach here is different than that normally used in two dimensions, we also present the two-dimensional case. The derivation of the three-dimensional genus allows us to make explicit what is meant by a hole in an object (e.g., the hole in a ring). We also give algorithms for computing the three-dimensional digital genus and discuss the importance of three-dimensional genus in contrast with the two-dimensional genus.

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A. D. BLOCH

Technical Information Officer

1. Introduction

The analysis of three-dimensional (3D) images has become of increasing interest (see, for example, [1-6]) with the rapid growth of computed tomography, in which discrete 3D representations of solid objects are reconstructed from sets of projections. This report, in which we define (and give algorithms for computing) the genus of a binary 3D image (as might be obtained by applying a threshold to an image produced by computed tomography), is one of a series [7-10] on the geometry and topology of 3D arrays.

Geometrical and topological properties of digital arrays play an important role in image analysis and recognition. While for two-dimensional (2D) images the genus of a digital image is not a particularly useful property per se, it is of theoretical interest. For example, there is a well developed theory [11,12] of the topological properties (e.g., connectedness, objects, holes) of 2D digital images which combine with various geometric concepts (e.g., arcs, curves) to form the basis of a consistent representation scheme that underlies many image analysis tasks such as object extraction and detection, thinning, and skeletonization.

We assume some familiarity with the notions of connectedness, objects, holes, etc., as in [11,12] for 2D digital images. Connectedness is a relation defined on binary valued points of a digital array which partitions the array into

objects (connected components of points with image value 1) and non-objects (connected components of points with image value 0), of which all but a single (unique) background component are holes. Simple algorithms may be given for labeling (and counting) objects and holes.

The genus of a 2D digital image may be defined to be the number of objects minus the number of holes in the image. While counting objects only or holes only is necessarily a global operation [13] there exist counting functions of local patterns of points which determine the genus of the image [11-15]. When 4-connectedness (see below, Section 3) is used for the 1's (a subset S) of the image, we have

$$G_4(S) = \psi_1 - \psi_2 + \psi_3 \quad (1)$$

where

ψ_1 = number of 1's in the image

ψ_2 = number of 1x2 blocks of 1's in the image (in all orientations)

ψ_3 = number of 2x2 blocks of 1's in the image

and when 8-connectedness is used for S we have

$$G_8(S) = \psi_1 - (\varphi_2 + \varphi_3) + \varphi_4 - \varphi_5 \quad (2)$$

where

ψ_1 = number of 1's in the image

φ_2 = number of times the pattern $\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}$ or $\begin{smallmatrix} 1 \\ & 1 \end{smallmatrix}$ occurs in the image

φ_3 = number of times the pattern $\begin{smallmatrix} & 1 \\ 1 & 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}$ occurs in the image

φ_4 = number of times the pattern $\begin{smallmatrix} & 1 & \\ 1 & 1 & \end{smallmatrix}$, $\begin{smallmatrix} & 1 & 1 \\ & 1 & \end{smallmatrix}$, $\begin{smallmatrix} 1 & & \\ & 1 & 1 \end{smallmatrix}$, or $\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}$ occurs in the image

φ_5 = number of times the pattern $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ occurs in the image

For 3D digital images we may define connectedness (see below, Section 4) as a relation on binary valued points of a digital array in a fashion analogous to the 2D definition, except that we call all but the background components of 0's cavities rather than holes [7,8]. The reason for this change in nomenclature will soon become evident.

In [15] it was argued that for 6-connected sets of 1's (see below) in a 3D digital image the analog of Eq. 1 is

$$G_6(S) = \psi_1 - \psi_2 + \psi_3 - \psi_4$$

where

ψ_1 = number of 1's in the image

ψ_2 = number of $1 \times 1 \times 2$ blocks of 1's in the image (in all orientations)

ψ_3 = number of $1 \times 2 \times 2$ blocks of 1's in the image (in all orientations)

ψ_4 = number of $2 \times 2 \times 2$ blocks of 1's in the image

and (as we shall see) the analog of Eq. 2 for 26-connected sets is

$$G_{26}(S) = \varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + \varphi_5 - \varphi_6 + \varphi_7 - \varphi_8$$

where the φ_i are as defined in Appendix A.

Consider the 3D array $\Sigma^{(3)}(x, y, z)$ constructed from a 2D array $\Sigma^{(2)}(x, y)$ by

$$\Sigma^{(3)}(x, y, z) = \begin{cases} \Sigma^{(2)}(x, y) & \text{if } z = z_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\Sigma^{(2)}$ has 0 objects and H holes. It is easy to see that with G_4 and G_8 defined on $\Sigma^{(2)}$, and G_6 and G_{26} defined on $\Sigma^{(3)}$, we have $G_4=G_6$ and $G_8=G_{26}$, and that the image $\Sigma^{(3)}$ has no cavities. Thus, although cavities are the 3D analog of 2D holes as defined by the connectedness relations, the functions G_6 and G_{26} do not give the number of 3D objects minus the number of 3D cavities. In the simple construction above, we see that G_6 and G_{26} are also affected by what are intuitively 3D holes, i.e., holes in a sheet.

In 3D the genus is sometimes defined [14,15] to be the number of objects minus the number of holes plus the number of cavities. Since for 3D digital images the topological concepts of object and cavity are well understood (in particular, algorithms similar to those for 2D images can be given for counting them) a definition of genus would define the number of holes, and conversely. What we desire is some evidence to support the assertion of either definition as being consistent with the corresponding ordinary topological notion.

Even in ordinary topology it is difficult to characterize holes. A hole may be thought of as a property of a boundary surface (separating binary regions) which makes it topologically equivalent to a torus; an object with H holes has a surface topologically equivalent to an H-holes torus. In another approach, an object is defined to have no holes if every simple closed curve in the object is continuously deformable within the object to a single point (see [7] where this approach

is used for digital images; for the present purposes we need a stronger statement, since we will in general be dealing with objects that may have many holes).

We see from these remarks that the concept of a hole is different from those of objects and cavities; we cannot point to or label the points which constitute a hole. Indeed, the points of objects and cavities cover the space, so that a hole is a property of these collections of points. Thus, when considering an object (and its cavities) we shall here try only to understand what is meant by the number of holes in the object, and not what is meant by a hole.

We begin by postulating the well-known Euler-Schlaefli relation as a fundamentally true statement of dimensional connectedness. This equation describes a relationship between the numbers of elements of different dimensionality in an interlinked system of these elements given the geometrical constraints of Euclidean space. This theory of interlinked systems of elements (structures) requires that these structures have, in effect, no holes (i.e., are singly connected). We have chosen this starting point because the process of extending the concepts of singly connected structures to more complex (multiply connected) structures furthers an understanding of what is meant by the definition of digital genus (number of holes), and not because multiply connected structures are not also well understood.

The approach taken here is constructive. That is, a structure is described by specifying the elements (of different dimensionalities) that constitute it. We proceed by making two associations: one from points of an image to structures, and one from components (objects, holes(2D), cavities(3D)) to elements of the structures. Even for 2D digital images, this approach is essentially different than the exposition of genus found in [11,12,13]. Thus, following the introduction of structures and the Euler-Schlaefli identity (Section 2) we illustrate this technique by presenting a derivation of genus for 2D digital images (Section 3) before developing the genus (number of holes) of a 3D digital image (Section 4). In Section 5, we give an algorithm for computing the 3D digital genus of an image.

Some early work on topology of 3D digital arrays is found in [14,15], and several theoretical papers [16,17] also consider generalizations to higher dimensions. More recently [18-20] discuss the topology of surfaces of 3D objects.

2. Space Structures

The theory of structures given here is from the excellent book by Loeb [21]. In general a structure is an underlying pattern of interaction between things. When a structure is confined to a Euclidean space, geometrical constraints are imposed on the structure, which allow us to write various quantitative expressions of spatial order. The well known Euler-Schlaefli equation

$$\sum_{i=0}^j (-1)^i N_i = 1 + (-1)^j$$

describes the relationship between the numbers of elements of different dimensionality in a multidimensional singly connected structure confined to Euclidean space. Here N_i is the number of elements of dimensionality i and j is the dimensionality of the structure. [In two dimensions we write this as the more familiar Euler invariant for any planar graph,

$$V-E+F = 2$$

where

$V = N_0$ = number of vertices

$E = N_1$ = number of edges

$F = N_2$ = number of faces.]

We note that in this identity the infinite region surrounding the structure is considered an element of dimensionality N_j .

Some remarks about the use of this identity are necessary. First, dimensionality refers to numbers of degrees of freedom.

A point confined to move inside a cube or sphere may move in three mutually perpendicular independent directions. A point confined to the surface of a cube or of a sphere (even though not flat) has only two independent directions of motion. Similarly, a line or curve has dimensionality 1 and a point has dimensionality zero.

Next, this identity applies only to structures that are singly connected. A structure is multiply connected if there are two or more curves between any two points which cannot be brought into alignment by continuous deformation (cf. the definition of the class of objects with no holes in [7]). This has a profound effect on the way we must view structures. Consider, for example, the 2D structure



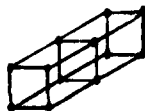
The singly connected restriction forces us to view the region surrounded by the vertices and edges as a face of the structure (the exterior is also considered a face). Just as an edge is commonly used to represent an independent relation between two vertices in a graph, the face in the above structure represents an independent relation between each vertex and the other two. In the 3D structure (tetrahedron)



we are left a choice: The (3D) structure consists of 4 vertices, 6 edges together with either 3 faces and 1 "background" cell

(structured element of dimensionality 3), so that the structure is cup-like, or 4 faces so as to create a surface that divides the three-dimensional space into an enclosed cell and all the space outside it.

Additionally, we shall require that the following closure postulate be satisfied: at least two edges join at every vertex, and at least two faces meet at every edge. Thus we exclude tree graphs from consideration as structures (although it is possible by adopting a peculiar vantage point to view tree graphs as structures [21, pp. 8, 9]), and in structures like that above the choice of structures is removed. Nevertheless, we are left with a choice in structures such as



Either there are ten faces and two cells or eleven faces and three cells.

3. 2D Digital Images

A 2D digital image Σ is a 2D lattice of points defined by pairs of Cartesian coordinates (x,y) which we may take to be integer valued. A binary 2D digital image is a 2D digital image whose points have values 0 or 1. The geometrical and topological concepts presented here may be found in [11,12].

We will consider two types of neighbors of a point $p=(x,y)$:

- (a) the neighbors (u,v) such that $|x-u| + |y-v| = 1$,
which we call edge neighbors of p ;
- (b) the neighbors (u,v) such that $|x-u| = 1$ and $|y-v| = 1$,
which we call corner neighbors of p .

The four edge neighbors of p are said to be 4-adjacent to p , and the eight edge and corner neighbors of p are said to be 8-adjacent to p .

A path from p to q is a sequence of points $p=p_0, \dots, p_n=q$ of Σ such that p_i is adjacent to p_{i-1} , $1 \leq i \leq n$, and any point alone is a path of length zero. In general, for any non-empty subset S of Σ we speak of (4 or 8)-paths in S depending on the type of adjacency used.

Let S be a subset of points of Σ and to avoid special cases assume that S does not meet the border of Σ . We say that p and q are connected in S if there is a path from p to q in S . Readily, connectivity is an equivalence relation, and the equivalence classes under this relation are called components (or objects) of S . Note that these are each two definitions in one, depending on the type of adjacency used, 4- or 8-.

Similarly we can consider components of the complement \bar{S} of S . Evidently exactly one of these components contains the border of Σ ; we call this component the background of S . All other components of \bar{S} (if any) are called holes in S . To avoid ambiguous situations, we will require that opposite types of connectedness be used for S and \bar{S} . Simple algorithms have been given to label and count components of S and \bar{S} [11].

Let A, B be subsets of Σ . We say that A surrounds B if any path from (a point of) B to (a point of) the border of Σ must meet (contain a point of) A . From [12] we have

Theorem 1. Any S surrounds its holes and is surrounded by its background.

Theorem 2. The adjacency graph of S (the graph whose nodes are components of S and \bar{S} and whose edges are defined by adjacency of components of S and \bar{S}) is a directed tree rooted at the background component of \bar{S} under the relation of surroundedness.

These two facts are significant in developing an algorithm to compute the genus of a set S (or \bar{S}) of a digital image. Namely, we can count the components of S as objects and count each set that is surrounded by a component of S as a hole to compute the genus. Thus it suffices to consider the genus of a single component of S and its complement.

With every point of a digital image we may associate a pixel (or "cell"), which is a square region defined by four corners and four edges which are shared with the pixels of neighboring points. We associate the binary value of each point with the

face of the pixel.

The genus $G(S)$ of a set is the number $O(S)$ of objects (components of S) minus the number $H(S)$ of holes in S . Thus, the component labeling and counting algorithms may be used to determine the genus of a set S . While the labeling/counting algorithms are global and sequential in nature, they have been used to demonstrate the existence of local parallel counting functions that determine the genus by an induction on the number of points in S [12]; given any n point object for which the local counting functions agree with the global counting algorithms, it can be shown that adding a point to the image will have the same effect on both.

We will see later that this approach cannot be used for 3D digital images. In 3D we can present algorithms to label and count objects (components of S) and cavities (non-background components of \bar{S}), but not holes; a hole is not something to which we can assign a label in the same sense that we label objects or cavities.

Consider the genus $G(\bar{S})$ of the complement \bar{S} of S in a 2D digital image. We see (from the adjacency tree) that every object in S is a hole in \bar{S} (i.e., $O(S) = H(\bar{S})$) and that every object of \bar{S} except the background is a hole in S (i.e., $O(\bar{S}) - 1 = H(S)$). We thus have

$$O(S) - H(S) = H(\bar{S}) - O(\bar{S}) + 1$$

or

$$G(S) + G(\bar{S}) = 1$$

Since we use opposite types of connectedness for S and \bar{S} , we see that the digital genus we define for each type of connectedness must satisfy $G_4(S) + G_8(\bar{S}) = 1$ and $G_8(S) + G_4(\bar{S}) = 1$.

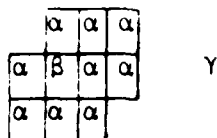
We first consider the genus of a single 8-component C&S. Consider the graph whose vertices are corners of pixels in S and whose edges are edges of these pixels, e.g., from the image

```

      1 1 1
    1   1 1
    1 1 1

```

we obtain the graph



(Clearly every such graph is connected and planar.) Viewing the graph as a structure we can identify (specify) three distinct types of faces:

- α) faces arising from pixels in C
- β) faces arising from components of \bar{C} surrounded by C
- γ) the "background" face arising from the component of \bar{C} not surrounded by C.

Clearly, every pixel of C becomes a face of the graph by construction. The single component of \bar{C} which surrounds C is associated with the "background" face of the structure. Every other component $D \subseteq \bar{C}$ is surrounded by C. Define an adjacency relation between the component D and the pixels of the D-border of C, and we see that we may associate each component D with the face of the structure adjacent to the faces arising from the pixels of the D-border of C. The face associated with each D is unique since the D-border of C surrounds D. In general faces of type (B) are

n-gons with $n \geq 4$. We can now write

$$V-E+F = V-E+(\#\alpha + \#\beta + \#\gamma) = 2$$

or

$$1-\#\beta = V-E+\#\alpha$$

where $\#\alpha$, $\#\beta$, $\#\gamma$ are the numbers of these types of faces in the structure.

Now $\#\beta$ is the number of faces resulting from components of \bar{C} surrounded by C , i.e., holes in C . As we have seen, it is sufficient to sum the number of holes in each $C \in S$ to determine the number of holes in S . Thus, if we sum V , E , and $\#\alpha$ over the structure of every component $C \in S$ we have

$$O(S)-H(S) = V-E+\#\alpha$$

where $O(S)$ is the number of objects and $H(S)$ is the number of holes in S .

We now consider computing $V-E+\#\alpha$ directly from the digital image Σ . Clearly $\#\alpha$ is just the number of pixels with image value 1. Similarly, it is not hard to see that

$$V = 4\psi_1 - 2\psi_2 - \psi_3 + \psi_4 - \psi_5$$

$$E = 4\psi_1 - \psi_2$$

so that

$$V-E+\#\alpha = \psi_1 - (\psi_2+\psi_3) + \psi_4 - \psi_5$$

where

ψ_1 = number of points in S

ψ_2 = number of times the pattern $\begin{smallmatrix} & 1 \\ 1 & 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 \\ 1 & 1 \end{smallmatrix}$ occurs in the image

ψ_3 = number of times the pattern $\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & \\ 1 & 1 \end{smallmatrix}$ occurs in the image

ψ_4 = number of times the pattern $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}$ occurs in the image

φ_5 = number of times the pattern $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ occurs in the image.

We thus have

$$G_8(S) = O(S) - H(S) = \varphi_1 - (\varphi_2 + \varphi_3) + \varphi_4 - \varphi_5$$

as the genus of S when we use 8-connectivity for S .

Several approaches suggest themselves for dealing with the 4-connected case. For example, we have already noted that $G(\bar{S}) + G(S) = 1$. Consider the structure of Σ as per the construction above. It is easy to see that for an $m \times n$ array of points we count V' , E' , and F' of this structure as

$$V' = (n+1)(m+1)$$

$$E' = 2mn + m + n$$

$$F' = nm + 1$$

so that

$$V' - E' + F' = 2$$

Counting directly from the image as before we can write

$$V' = 4\varphi_1 - 2\varphi_2 - \varphi_3 + \varphi_4 - \varphi_5 + \psi_3 + 2m + 2n$$

$$E' = 4\varphi_1 - \varphi_2 + \psi_2 + 2m + 2n$$

$$F' = \varphi_1 + \psi_1 + 1$$

so that

$$V' - E' + F' = [\varphi_1 - (\varphi_2 + \varphi_3) + \varphi_4 - \varphi_5] + [\psi_1 - \psi_2 + \psi_3] + 1 = 2$$

where

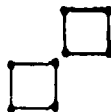
ψ_1 = number of points in \bar{S}

ψ_2 = number of times the pattern $\begin{smallmatrix} 0 \\ 0 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ occurs in the image

ψ_3 = number of times the pattern $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ occurs in the image

Thus, we have $G_8(S) + [\psi_1 - \psi_2 + \psi_3] = 1$, so that $G_4(\bar{S}) = \psi_1 - \psi_2 + \psi_3$. We shall see that for 3D digital images this approach requires an additional assumption.

Thus, let us proceed as before by constructing a structure which corresponds to a 4-component C of S . The construction is similar except that for the configuration $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ (and its reflection) we construct the subgraph



i.e., there are two vertices at the center of the configuration. The proof that we can identify three distinct types of faces of the structure $(\alpha-\gamma)$ stands without change, and we obtain

$$O(S) - H(S) = V - E + \# \alpha$$

We can count V and E directly from the image as

$$V = 4\psi_1 - 2\psi_2 + \psi_3$$

$$E = 4\psi_1 - \psi_2$$

$$\# \alpha = \psi_1$$

so that

$$V - E + \# \alpha = \psi_1 - \psi_2 + \psi_3$$

where the ψ_i are as before but for image values of 1. We thus have

$$G_4(S) = O(S) - H(S) = \psi_1 - \psi_2 + \psi_3$$

Even with the complication of the additional vertex for a pair of diagonal 1's the calculations of the preceding paragraph stand without change, so that we could in principle define the genus $G_8(\bar{S})$ of the complement in terms of $G_4(S)$ by considering the structure corresponding to Σ .

Before discussing the genus of a 3D digital image in the next section we should point out that the structures used here for 2D digital images are by no means the only structures we can use

for defining genus. For example, it is not hard to see that $G_4 = \psi_1 - \psi_2 + \psi_3$ may be interpreted as counting vertices, edges, and faces on a graph whose vertices are points of the image and whose edges are defined by 4-connectivity, although a somewhat different association must be made between faces of the graph and subsets of the digital image. Note that when the edges of such a graph are defined by 8-connectivity the "structure" becomes three-dimensional. For 3D digital images this kind of graph will lead to an efficient algorithm for computing the genus.

4. Digital Binary Images in Three Dimensions

A three-dimensional digital image Σ is a three-dimensional lattice of points defined by triples of Cartesian coordinates (x,y,z) which we may take to be integer valued. A binary 3D digital image is a digital image whose points have values 0 or 1.

In this section we review the concepts of adjacency, connectedness, and components of points in Σ as developed in [7,8,10].

We will consider two types of neighbors of a point $p = (x,y,z)$:

(a) the neighbors (u,v,w) such that $|x-u| + |y-v| + |z-w| = 1$, which we call 6-neighbors of p

(b) the neighbors (u,v,w) such that $\max[|x-u|, |y-v|, |z-w|] = 1$, which we call the 26-neighbors of p .

The 6-neighbors are said to be 6-adjacent to p , and the 26-neighbors 26-adjacent to p .

A path from p to q in Σ is a sequence of points $p=p_0, \dots, p_n=q$ of Σ such that p_i is adjacent to p_{i-1} , $1 \leq i \leq n$. Any point alone is a path of length zero. In general, for any non-empty subset S of Σ we speak of (6- or 26-) paths in S depending on the type of adjacency used.

Let S be a subset of points of Σ , and to avoid special cases assume that S does not meet the border of Σ . We say that p and q are connected in S if there is a path from p to q in S . Readily, connectivity is an equivalence relation, and the equivalence classes under this relation are called components of S (or objects). Note that these are two definitions in one, depending on the type of adjacency used.

Similarly, we can consider components of the complement \bar{S} of S . Evidently exactly one of these components contains the border of Σ ; we call this component the background of S . All other components of \bar{S} (if any) are called cavities in S . To avoid ambiguous situations we will require that opposite types of adjacency be used for S and \bar{S} . Simple algorithms can be given for labeling and counting distinct components of S and \bar{S} [15,7].

The genus $G(S)$ of a set S in a 3D digital image is the number $O(S)$ of objects plus the number $C(S)$ of cavities in S minus the number $H(S)$ of holes in S . As already mentioned the definition of hole is not simple, and in particular holes cannot be labeled to facilitate counting them, as can be done with objects and cavities.

Let A, B be subsets of Σ . We say that A surrounds B if any path from (a point of) B to (a point of) the border of Σ must meet (contain a point of) A . As in 2D we have [7]

Theorem 3. Any S surrounds its cavities and is surrounded by its background.

Theorem 4. The adjacency graph of S (the graph whose nodes are components of S and \bar{S} and whose edges are defined by adjacency of components of S and \bar{S}) is a directed tree rooted at the background component of \bar{S} under the relation of surroundedness.

With every point of a digital image we may associate a voxel, usually a cubic region defined by 8 corners, twelve edges, and six faces which are shared with the voxels of neighboring points. We thus associate the binary value of each point with the cell (or volume) of the voxel.

For any 3D structure the Euler-Schlaefli equation is

$$V-E+F-Q = 0$$

where

V = number of vertices

E = number of edges

F = number of faces

Q = number of cells (including the background).

From a 3D digital image we may construct a structure from any single 27-component BCS whose vertices are corners of voxels with image value 1, whose edges are edges of these voxels, and whose faces are faces of these voxels. In analogy with the 2D development we can identify three types of cells:

(α') cells arising from voxels of B

(β') cells arising from components of \bar{B} surrounded by B

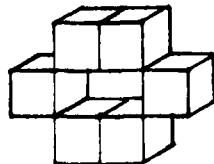
(δ') the "background" cell arising from the component of \bar{B} not surrounded by B.

In general, cells of type β' are polyhedrons with eight or more vertices. The details of this association are similar to those in the 2D proof.

This done, we find that the resulting structure is not necessarily singly connected. As a simple example consider the structure resulting from the digital image

```
0 1 1 0
1 0 0 1
0 1 1 0
```

which is



Some additional structural elements are required to make this structure singly connected (cf. the "missing faces" of the 2D development above).

In general, for the structure derived as above from any digital component $B \subseteq S$ we may compute $V-E+F$ ($\# \alpha + \# \beta + \# \gamma$) although this sum need not equal zero since the structure need not be singly connected. We define the number of holes in the object as the amount by which this sum differs from zero,

$$H(B) \equiv -V + E - F + (\# \alpha + \# \beta + \# \gamma)$$

Noting that $\# \beta$ is the number of cavities $C(B)$ in any component $B \subseteq S$ and that $\# \gamma = 1$ (for the background) we have

$$1 + C(B) - H(B) = V - E + F - \# \alpha$$

so that we may sum $V, E, F, \# \alpha$ over every component $B \subseteq S$ in the image to obtain

$$O(S) - H(S) + C(S) = V - E + F - \# \alpha$$

where $O(S)$ is the number of objects in the image, $C(S)$ is the total number of cavities in these objects, and $H(S)$ is the total number of holes in these objects. [We are tacitly assuming that $H(S) = \sum H(B)$, where the summation is over the components B of S . This additive property of holes of components is justified by the fact [14] that genus is a local set property, viz. $G(A) + G(B) = G(A \cap B) + G(A \cup B)$ for objects A and B .]

Remark: The definition of number of holes above may be related to the Euler-Poincare characteristic for finite complexes [22]. The structure corresponding to a digital component is a complex $K-L$ where L is a subcomplex of a simplicial complex K . Intuitively,

the sum $V-E+F-(\alpha+\beta+\gamma)$ differs from zero by some amount $v-e+f-g$, where these are computed on $K-L$ and L , respectively. Viewed in this fashion the number of holes is the sum $v-e+f-g$ for any structural elements added to the structure resulting from a digital image which will make it singly connected.

We now consider computing $V-E+F-\alpha$ directly from the digital image. Although more laborious than the 2D case, we can show that

$$G_{27}(S) = O(S) - H(S) + C(S) = \varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + \varphi_5 - \varphi_6 + \varphi_7 - \varphi_8$$

where

$$\varphi_1 = \#[1]$$

$$\varphi_2 = \#[2] + \#[3] + \#[4]$$

$$\varphi_3 = \#[5] + \#[6] + \#[7]$$

$$\varphi_4 = \#[8] + \#[9] + \#[10] + \#[11] + \#[12] + \#[13] + \#[14]$$

$$\varphi_5 = \#[15] + \#[16] + \#[17]$$

$$\varphi_6 = \#[18] + \#[19] + \#[20]$$

$$\varphi_7 = \#[21]$$

$$\varphi_8 = \#[22]$$

and by $\#[n]$ we mean the number of times the configuration n of Appendix A occurs in the picture (in all orientations).

As in the 2D case one way of proceeding to the 6-connected case is to note that for the structure corresponding to Σ we have $V'-E'+F'-Q' = 0$, and that counting directly from the image we have

$$V'-E'+F'-Q' = (\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + \varphi_5 - \varphi_6 + \varphi_7 - \varphi_8) - \psi_1 + \psi_2 - \psi_3 + \psi_4 + 1 = 0$$

or

$$(\psi_1 - \psi_2 + \psi_3 - \psi_4) - G_{27}(S) = 1$$

where the ψ_i count patterns of 0's as follows:

$$\psi_1 = \# [1]$$

$$\psi_2 = \# [2]$$

$$\psi_3 = \# [8]$$

$$\psi_4 = \# [20]$$

Now, every object of S is a cavity in \bar{S} (i.e., $O(S) = C(\bar{S})$) and every object of \bar{S} except the background is a cavity of S (i.e., $O(\bar{S}) - 1 = C(S)$). If we make the assumption that $H(S) = H(\bar{S})$ we have $O(S) + C(S) - H(S) = C(\bar{S}) + O(\bar{S}) - 1 - H(\bar{S})$ or $G(\bar{S}) - G(S) = 1$. Thus from the above equation we can define $G_6(\bar{S}) = \psi_1 - \psi_2 + \psi_3 - \psi_4$. Because we need this additional assumption, we consider it safer to base the definition of genus for 6-connectivity on a structure corresponding to a 6-connected object.

The construction of the structure for a 6-connected object is similar to that above, except that for configurations such as $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ and

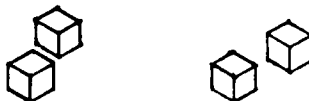
1st plane

$\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}$

2nd plane

$\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$

we use the structures



so that each 6-component forms a distinct structure. (We are glossing over the details of this construction; the essential point is that a vertex is not shared unless the voxels are connected inside a $2 \times 2 \times 2$ neighborhood containing the two voxels.)

Again, we identify some of the cells of the structure with components of the complement, and define the number of holes as the amount by which the sum $V-E+F-(\#\alpha+\#\beta+\#\gamma)$ differs from zero (or as the net contribution to the sum $V-E+F-Q$ needed to make the structure simply connected i.e., $V-E+F-(\#\alpha+\#\beta+\#\gamma)+H = 0$). For any single component $B \subseteq S$ we have

$$1+C(B) - H(B) = V-E+F-\#\alpha$$

so that for $O(S)$ components of a digital image we can determine

$$O(S)-H(S)+C(S) = V-E+F-\#\alpha$$

by summing V, E, F and $\#\alpha$ over the entire image. Counting directly from the image we find

$$V = 8\psi_1 - 4\psi_2 + 2\psi_3 - \psi_4$$

$$E = 12\psi_1 - 4\psi_2 + \psi_3$$

$$F = 6\psi_1 - \psi_2$$

$$\#\alpha = \psi_1$$

so that

$$G_6(S) = O(S) - H(S) + C(S) = \psi_1 - \psi_2 + \psi_3 - \psi_4$$

where the ψ_i are defined as before but for patterns of 1's in Σ .

Once again, we find that (with the additional assumption $H(S) = H(\bar{S})$) we can in principle define the genus of the complement in terms of $G_6(S)$ by considering the structure corresponding to Σ .

From the definitions of G_{27} and G_6 based on structure we see that $G_{27}(\bar{S}) - G_6(S) = 1$ and $G_6(\bar{S}) - G_{27}(S) = 1$ from counting elements in the structure corresponding to Σ . Thus, we have shown that $H(S) = H(\bar{S})$ since $O(S) = C(\bar{S})$ and $O(\bar{S}) - 1 = C(S)$.

5. An Algorithm to Compute $G(S)$

The equations

$$G_6(S) = \psi_1 - \psi_2 + \psi_3 - \psi_4$$

$$G_{27}(S) = \varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + \varphi_5 - \varphi_6 + \varphi_7 - \varphi_8$$

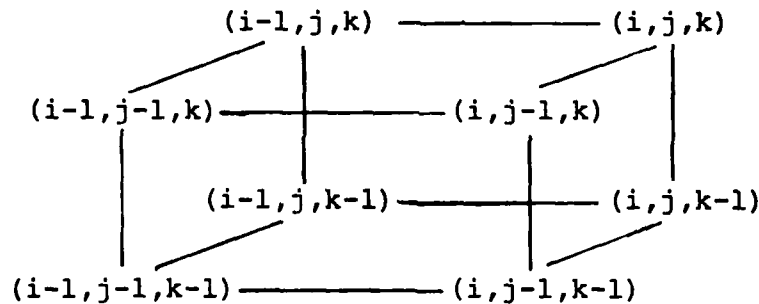
define the genus of a 3D digital image but are not algorithms for computing the genus. An algorithm for computing $G_6(S)$ is not difficult to devise, since the patterns of points of S for ψ_1 and ψ_4 occur in an orientation independent fashion and the patterns for ψ_2 and ψ_3 occur in three different orientations each. Thus it is sufficient to test for occurrences of nine distinct patterns in the digital image. For $G_{27}(S)$, however, there are 22 distinct patterns, many of which occur in many orientations.

In this section we give an algorithm for computing $G_{27}(S)$. The development of this algorithm is understood in terms of a graph defined by the 27-connectivity relation on the points of S in Σ . We point out that the graph used here is not similar to the structures used to define the genus, but is analogous to the alternative structure for 2D images mentioned at the end of Section 3. The φ_i may be given the following interpretation. Consider a graph $\langle S, E(S) \rangle$ whose vertices are points of S and whose edges are defined by the 26-connectivity relation. Then φ_1 is the number of vertices and φ_2 the number of edges of $\langle S, E(S) \rangle$. In general, since every point in a $2 \times 2 \times 2$ cube is 26-adjacent to every other point in the $2 \times 2 \times 2$

cube, φ_ℓ is the number of distinct ℓ -cliques in $\langle S, E(S) \rangle$ that are contained in a $2 \times 2 \times 2$ cube of points on the lattice.

This interpretation of the φ_ℓ leads to a simple and efficient algorithm for determining the φ_ℓ and thus $G_{26}(S)$. The following description of the algorithm is most easily understood in terms of using registers to count the contribution to each φ_ℓ of each $2 \times 2 \times 2$ neighborhood during a raster scan of Σ . However, the algorithm may be executed in parallel.

Let $K_{i,j,k}$ be the $2 \times 2 \times 2$ cube (during a raster scan with increasing values of i, j, k)



Let n be the number of 1's in $K_{i,j,k}$. By definition, φ_ℓ is the number of distinct objects consisting of ℓ points. Thus $\varphi_\ell(K_{i,j,k}) = \binom{n}{\ell}$. When S extends outside of $K_{i,j,k}$, and the φ_ℓ are applied to other $2 \times 2 \times 2$ cubes, some of the ℓ -point objects may be counted multiple times.

Let

$$\begin{aligned} n_1 &= \# [2] (\{ (i-1, j, k), (i-1, j-1, k), (i-1, j, k-1), (i-1, j-1, k-1) \}) \\ &\equiv \# [2] (K_{i,j,k}^1) \end{aligned}$$

$$\begin{aligned}
n_2 &= \# [2] (\{ (i-1, j-1, k), (i, j-1, k), (i-1, j-1, k-1), (i, j-1, k-1) \}) \\
&\equiv \# [2] (K_{i,j,k}^2) \\
n_3 &= \# [2] (\{ (i-1, j, k-1), (i, j, k-1), (i-1, j-1, k-1), (i, j-1, k-1) \}) \\
&\equiv \# [2] (K_{i,j,k}^3) \\
n_{12} &= \# [2] (\{ (i-1, j-1, k), (i-1, j-1, k-1) \}) \equiv \# [2] (K_{i,j,k}^{12}) \\
n_{13} &= \# [2] (\{ (i-1, j, k-1), (i-1, j-1, k-1) \}) \equiv \# [2] (K_{i,j,k}^{13}) \\
n_{23} &= \# [2] (\{ (i, j-1, k-1), (i-1, j-1, k-1) \}) \equiv \# [2] (K_{i,j,k}^{23}) \\
n_{123} &= \# [2] (\{ (i-1, j-1, k-1) \}) \equiv \# [2] (K_{i,j,k}^{123})
\end{aligned}$$

where $\# [n] (X)$ refers to the number of times the pattern n of Appendix A (in all orientations) occurs in X .

We can now solve for the net contribution $\Delta \varphi_{\ell} (K_{i,j,k})$ of $K_{i,j,k}$ to each $\varphi_{\ell} (S)$. Clearly the ℓ point objects in $K_{i,j,k}^1$ are counted in $K_{i-1,j,k}$, the ℓ point objects in $K_{i,j,k}^2$ are counted in $K_{i,j-1,k}$, and the ℓ point objects in $K_{i,j,k}^3$ are counted in $K_{i,j,k-1}$. Thus the count $\binom{n}{\ell}$ must be adjusted to

$$\binom{n}{\ell} - \binom{n_1}{\ell} - \binom{n_2}{\ell} - \binom{n_3}{\ell}$$

But now the numbers of ℓ point objects in $K_{i,j,k}^{12}$, $K_{i,j,k}^{13}$, and $K_{i,j,k}^{23}$ have been subtracted twice, so this must be further adjusted to

$$\binom{n}{\ell} - \binom{n_1}{\ell} - \binom{n_2}{\ell} - \binom{n_3}{\ell} + \binom{n_{12}}{\ell} + \binom{n_{13}}{\ell} + \binom{n_{23}}{\ell}$$

Finally, the ℓ point objects in $K_{i,j,k}^{123}$ have now been counted one too many times, so the final adjustment is

$$\Delta \varphi_{\ell} (K_{i,j,k}) = \binom{n}{\ell} - \binom{n_1}{\ell} - \binom{n_2}{\ell} - \binom{n_3}{\ell} + \binom{n_{12}}{\ell} + \binom{n_{13}}{\ell} + \binom{n_{23}}{\ell} - \binom{n_{123}}{\ell}$$

This net contribution of each $2 \times 2 \times 2$ cube may be computed for each cube of the image in parallel and summed.

Similarly this gives a simple expression for the net contribution $\Delta G_{26}(K_{i,j,k})$ of $K_{i,j,k}$ to $G_{26}(S)$ which may be computed for each $K_{i,j,k}$ in parallel and summed:

$$\Delta G_{26}(K_{i,j,k}) = \sum_{\ell=1}^n (-1)^{\ell-1} \left[\binom{n}{\ell} - \binom{n}{\ell} n_1 - \binom{n}{\ell} n_2 - \binom{n}{\ell} n_3 + \binom{n}{\ell} n_{12} + \binom{n}{\ell} n_{13} + \binom{n}{\ell} n_{23} - \binom{n}{\ell} n_{123} \right]$$

Since the expression inside the brackets requires a significant amount of computation, we suggest the use of a lookup table as follows. From a $2 \times 2 \times 2$ cube of binary points form an 8-bit number t . Clearly the values of $n, n_1, n_2, n_3, n_{12}, n_{13}, n_{23}$, and n_{123} may be predetermined for every $0 \leq t \leq 255$ and the values of $\Delta G_{26}(\langle t \rangle)$ stored in a table, where $\langle t \rangle$ is the $2 \times 2 \times 2$ cube of points whose 8-bit number is t . In Appendix B we list the values of $\Delta G_{26}(\langle t \rangle)$ for one method of forming the 8-bit number t from the 8! different ways of doing this. It should be noted that in this table $-2 \leq \Delta G_{26}(\langle t \rangle) \leq 1$, and that most of the entries are zeros, so that if necessary the size of the table may be significantly reduced.

Of course, there is another alternative to computing the genus of a 27-connected subset S : Namely, use the simpler 6-connected algorithm to determine the genus of \bar{S} , and employ the identity $G(\bar{S}) - G(S) = 1$.

6. Conclusion and Remarks

We have proposed definitions of digital image subsets based on the association of a structure with each component of the subset. For 2D digital image components we found that these structures were singly connected, so that by straightforward application of the Euler-Schlaefli identity we could compute the genus of a collection of components. While for 3D digital components the structures are not necessarily singly connected we still began with the Euler-Schlaefli equation in order to give insight as to the nature of holes in 3D images. A definition of the number of holes in a subset of a digital image is equivalent to defining the genus of the subset since the numbers of objects and cavities are well understood. We were thus able to define, and give an algorithm for computing, the genus of 3D digital image subsets.

We noted in the introduction that the 2D genus is not a particularly useful property, particularly since it is easy to determine the number of objects and number of holes independently. In contrast to the mostly theoretical interest in 2D genus, the 3D genus is of more practical value. Since the numbers of objects and cavities of a 3D image may be determined independently, the genus may be used to determine the number of holes in an object or objects. Clearly the number of holes in an object is an important descriptor that should be useful in many object recognition tasks.

We have already noted that the structures we have chosen are not unique. While the alternative "structures" mentioned

in Sections 3 and 5 are not strictly structures as defined here, we remind the reader that from a peculiar vantage point [21, pp.8-9] these too may be viewed as structures to which the Euler-Schlaefli equation is applicable. We shall not pursue these alternative structures further.

Another question entirely concerns the use of 6- and 26-connectivity exclusively as complementary types of connectedness. For example, we may define 18-connectedness between points whose voxels are face-adjacent or edge-adjacent [see 7]. Clearly this relation is an equivalence relation whose equivalence classes define components, and we can certainly construct structures analogous to those made for 6- and 26-components. We thus see no obstacle to the use of 6- and 18-connectivity as complementary types in a 3D digital image. We have chosen not to pursue this here since the use of 6- and 26-connectedness appears more natural (e.g., the only natural distance metrics on a 3D lattice correspond to the number of points on a shortest 6- or 26-path between two points [7]). However, 18-connectivity appears to be equivalent to the connectedness underlying the surface topology of [18,19].

We may also scrutinize the use of structures to model the image space. For example, in computed tomography a 3D digital image is a representation of a 3D continuous image space of objects which are not polyhedral structures as used here. In Appendix C we explore the relationship between G_{26} and G_6 of a digital image to the genus of a binary region of continuous space. This analysis indicates that if certain digitization

requirements are met by the digital representation then G_{26} and G_6 in fact determine the genus of the continuous space, and that these digitization criteria may be satisfied by using a fine enough resolution.

Finally, in Sections 4 and 5 we introduced a graph theoretic interpretation of the genus of subsets of \mathbb{Z} , whereby an object is represented by a graph $\langle S, E(s) \rangle$ on the lattice \mathbb{Z} whose edges are defined by the appropriate adjacency relation between pairs of points of S . Thus, we may view G_4 , G_8 , G_6 , G_{26} as functions of the appropriate graph.

We noted earlier that genus is a linear set property, viz.

$$G(A) + G(B) = G(A \cap B) + G(A \cup B)$$

(see [14] for a proof of this). Expanding our notation to $G(S) = G(\langle S, E(S) \rangle)$ we have

$$G(A \cap B) = G(\langle A \cup B, E(A \cup B) \rangle) - G(\langle A, E(A) \rangle) - G(\langle B, E(B) \rangle)$$

from which we see that

$$G(A \cap B) = G(\langle \emptyset, E(A \cup B) - E(A) - E(B) \rangle)$$

where \emptyset is the empty set. This is similar to the definitions in [8] of simple closed curves and surfaces by sets of ordered pairs (p, q) of points $p \in S$, $q \in \bar{S}$, since these pairs specify the edges of $E(A \cup B) - E(A) - E(B)$. This approach deserves further investigation.

Appendix A

The 256 binary assignments to a $2 \times 2 \times 2$ neighborhood of points in a 3D image may be grouped by symmetry into 22 distinct patterns. The pictorial display here, taken from [14], shows voxels with image value 1 as cubes. In Sections 4 and 5 the notation $\#[n]$ refers to the numbering here of these patterns. In patterns 15-22 only all hidden voxels have value 1.

(All zeros)



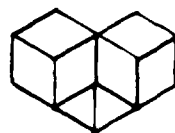
1 ($=\overline{22}$)



2 ($=\overline{21}$)



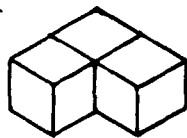
3 ($=\overline{20}$)



4 ($=\overline{19}$)



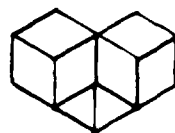
5 ($=\overline{18}$)



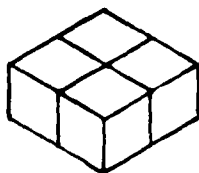
6 ($=\overline{17}$)



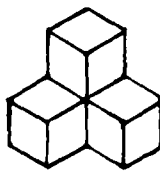
7 ($=\overline{16}$)



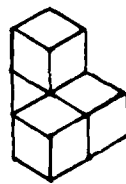
8 ($=\overline{15}$)



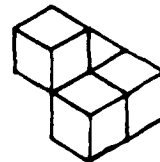
9 ($=\overline{9}$)



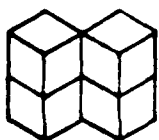
10 ($=\overline{10}$)



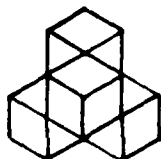
11 ($=\overline{11}$)



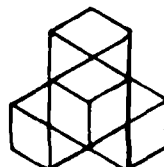
12 ($=\overline{12}$)



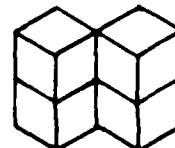
13 ($=\overline{13}$)



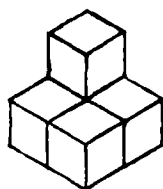
14 ($=\overline{14}$)



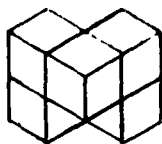
15 ($=\overline{8}$)



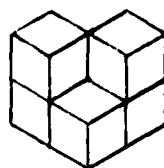
16 ($=\overline{7}$)



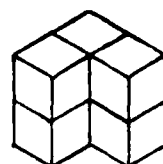
17 ($=\bar{6}$)



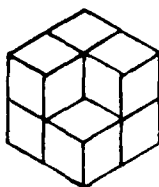
18 ($=\bar{5}$)



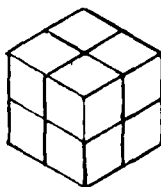
19 ($=\bar{4}$)



20 ($=\bar{3}$)



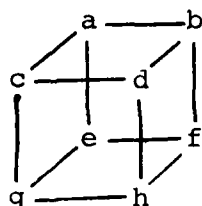
21 ($=\bar{2}$)



22 ($=\bar{1}$)

Appendix B

The values of the net contribution $\Delta G_{26}(K_{i,j,k})$ of a $2 \times 2 \times 2$ cube $K_{i,j,k}$ to the genus $G_{26}(S)$ of a subset S of a digital image are tabulated below by a decimal index t . The binary representation of the number t is formed from the $2 \times 2 \times 2$ cube of binary valued points



as the bit string $t=abcdefgh$. The point with binary value b is considered to be (i,j,k) .

Note that only 49 entries for $\Delta G_{26}(<t>)$ are non-zero, and that $-2 \leq G_{26}(<t>) \leq 1$, so that a significant amount of compression may be achieved if the table size is a problem.

<u>t</u>	<u>$\Delta G(<t>)$</u>	<u>t</u>	<u>$\Delta G(<t>)$</u>
0	0	17	0
1	0	18	0
2	1	19	0
3	0	20	0
4	0	21	0
5	0	22	0
6	0	23	0
7	0	24	-1
8	0	25	-1
9	-1	26	-1
10	0	27	-1
11	-1	28	0
12	0	29	0
13	0	30	0
14	0	31	0
15	0	32	0
16	0	33	-1

<u>t</u>	<u>$\Delta G(<t>)$</u>	<u>t</u>	<u>$\Delta G(<t>)$</u>
34	0	89	0
35	-1	90	0
36	-1	91	0
37	-1	92	0
38	-1	93	0
39	-1	94	0
40	-1	95	0
41	-2	96	0
42	-1	97	0
43	-2	98	0
44	-1	99	0
45	-1	100	0
46	-1	101	0
47	-1	102	0
48	0	103	0
49	0	104	0
50	0	105	0
51	0	106	0
52	0	107	0
53	0	108	0
54	0	109	0
55	0	110	0
56	-1	111	0
57	-1	112	0
58	-1	113	0
59	-1	114	0
60	0	115	0
61	0	116	0
62	0	117	0
63	0	118	0
64	0	119	0
65	0	120	0
66	0	121	0
67	0	122	0
68	0	123	0
69	0	124	0
70	0	125	0
71	0	126	0
72	0	127	0
73	0	128	0
74	0	129	-1
75	0	130	0
76	0	131	-1
77	0	132	0
78	0	133	0
79	0	134	0
80	0	135	0
81	0	136	0
82	0	137	-1
83	0	138	0
84	0	139	-1
85	0	140	0
86	0	141	0
87	0	142	0
88	0	143	0

<u>t</u>	<u>$\Delta G(<t>)$</u>
144	0
145	0
146	0
147	0
148	1
149	1
150	1
151	1
152	0
153	0
154	0
155	0
156	1
157	1
158	1
159	1
160	0
161	-1
162	0
163	-1
164	0
165	0
166	0
167	0
168	0
169	-1
170	0
171	-1
172	0
173	0
174	0
175	0
176	0
177	0
178	0
179	0
180	1
181	1
182	1
183	1
184	0
185	0
186	0
187	0
188	1
189	1
190	1
191	1
192	0
193	0
194	0
195	0
196	0
197	0
198	0
199	0

<u>t</u>	<u>$\Delta G(<t>)$</u>
200	0
201	0
202	0
203	0
204	0
205	0
206	0
207	0
208	0
209	0
210	0
211	0
212	0
213	0
214	0
215	0
216	0
217	0
218	0
219	0
220	0
221	0
222	0
223	0
224	0
225	0
226	0
227	0
228	0
229	0
230	0
231	0
232	0
233	0
234	0
235	0
236	0
237	0
238	0
239	0
240	0
241	0
242	0
243	0
244	0
245	0
246	0
247	0
248	0
249	0
250	0
251	0
252	0
253	0
254	0
255	0

Appendix C

In this appendix we outline an analysis of the relationship between the genus of a digital image and the continuous space it represents. Let Σ^* be a continuous space composed of binary regions; the union of all regions with value 1 we call S^* , and the union of the others \bar{S}^* . We assume that the background is a region of \bar{S}^* .

We wish to state the circumstances under which a digital image Σ is a representation of Σ^* that preserves the topology of Σ^* . These circumstances take the form of a local digitization requirement that we impose on Σ . Let $N(p)$ be the 8-neighbors of $p \in \Sigma$ including p when Σ is two-dimensional, and let $N(p)$ be the 26-neighbors of $p \in \Sigma$ including p when Σ is three-dimensional. Associated with p and $N(p)$ in Σ^* we have p^* and $N^*(p)$.

First, we will require every local binary region of Σ^* to have a representation in Σ :

$$(i) \quad O(S^* \cap N^*(p)) = O(S \cap N(p))$$

$$(ii) \quad O(\bar{S}^* \cap N^*(p)) = O(\bar{S} \cap N(p))$$

For the case where Σ is two-dimensional this is all we need. Let ϕ be either G_4 or G_8 . By the definition of ϕ (Section 3) it is not hard to see that for $p \in S$

$$\phi(S) = \phi(S - \{p\}) + \phi(S \cap N(p)) - \phi([S \cap N(p)] - \{p\})$$

Now every n point object S can be constructed from an $n-1$ point object $S - \{p\}$ with p the eastmost of the northmost points of S .

We note that when $S=\{p\}$, $\phi(S)=1$ and $G(S^*)=1$ by (i)-(ii), and proceed by induction. By induction hypothesis $G((S-\{p\})^*) = \phi(S-\{p\})$, and since $p \in S$, $G([S^* \cap N^*(p)] - p^*) = \phi([S \cap N(p)] - \{p\})$. Now $G(S^* \cap N^*(p)) = O(S^* \cap N^*(p)) - H(S^* \cap N^*(p))$ and by (ii) we must have $H(S^* \cap N^*(p)) = H(S \cap N(p))$ since $H(S \cap N(p)) = 0$. That $G(S^* \cap N^*(p)) = O(S^* \cap N^*(p)) = \phi(S \cap N(p))$ is easily verified by using a program to enumerate all possible configurations of an eastmost, northmost point's neighborhood.

The case where Σ is three-dimensional requires an additional digitization requirement which ensures that any hole occurring in a small enough neighborhood of a point in the continuous image exhibits itself in a well defined way in the digital image. The form of this third requirement is similar to the definition of the class of objects with no holes suggested in [7].

We define a closed path $\pi = p_1, \dots, p_m$ as one with $p_1 = p_m$. A point p_i of π is a path end if it has exactly one neighbor that is also a point of π , and is a path corner if it has exactly two neighbors on π which have a common neighbor q other than π_i . Two paths $\pi = p_1, \dots, p_m$ and $\pi' = p'_1, \dots, p'_n$ that lie in $C \subseteq S$ are strongly equivalent in the component C if

- a) There exists a path end $p_i \in \pi$ such that π' is the same as π with p_i and the repetition of p_{i+1} deleted.
- b) There exists a corner $p_i \in \pi$ such that π' is the same as π with p_i replaced by q , the common neighbor of p_{i-1} and p_{i+1} (so that $m=n$)

(If p_i occurs several times on π we delete or divert all instances of it). We call the reflexive, symmetric, transitive closure of strong equivalence equivalence in C . We say that π is reducible in C if it is equivalent in C to a degenerate closed path consisting of a single point. [This definition of equivalence of closed paths is motivated by the concept of continuously deformable simple closed curves in continuous space].

We can now state our third requirement as

- (iii) If $S^* \cap N^*(p)$ has a hole then there exists a closed path in $S \cap N(p)$ that is irreducible in $S \cap N(p)$

Intuitively, this means that if there is a simple closed curve in $S^* \cap N^*(p)$ which is not continuously deformable to a single point without leaving $S^* \cap N^*(p)$, then a similar phenomenon should occur in $S \cap N(p)$.

The inductive argument that $G(S^*) = \phi(S)$ where ϕ is either G_{26} or G_6 is similar to that of the 2D case. For $G_{26}(S)$ it is easy to see that since $p \in S$, every path in $S \cap N(p)$ is reducible to p so that $H(S^* \cap N^*(p)) = 0$, and that $C(S^* \cap N^*(p)) = 0$. It is easily verified (by writing a program to test each configuration) that for an eastmost, northmost, topmost point of S , $C(S \cap N(p)) = G_{26}(S \cap N(p))$. For the 6-connected case it is simpler to show by induction that $G_6(\bar{S}) - G_{26}(S) = 1$ (or $G_{26}(\bar{S}) - G_6(S) = 1$).

We summarize these results in

Proposition C.1. If $S \subseteq \Sigma^*$ meeting requirements (i) and (ii) if Σ is two-dimensional, and meeting requirements (i)-(iii) if

Σ is three-dimensional, then $G(S) = G(S^*)$ and $G(\bar{S}) = G(\bar{S}^*)$.

Evidently, (i) and (ii) may be achieved through the use of a fine enough resolution. While this is not as evident for (iii), we notice that we could choose a resolution so fine that $H(S^* \cap N^*(p)) = 0$ so that (iii) becomes vacuous.

A second interpretation may be given to (i)-(iii); if we define the genus of a digital picture as in Sections 3 and 4, then (i)-(iii) define the classes of (2D and 3D) continuous image spaces whose topology is preserved in their digital representation.

We leave a rigorous analysis of this relationship as an open problem.

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